JRAHS Ext 1 T2 2007

QUESTION 1 (9 Marks) **Marks**

(a) (i) How many ways could the letters of the word SOCIETY be arranged if each arrangement begins with C and ends with E. **1**

(ii) If an arrangement is selected at random, find the probability that it contains the word SOY. **1**

(b) The displacement function of a particle moving x metres along a straight line after t seconds is given by $x = \sqrt{2}\cos 5t - \sin 5t$. Show that its acceleration function is of the form $\ddot{x} = -n^2 x$ and find the value of n. **2**

(c) A plane travelling at a constant height of 1500 metres at a speed of 600 *km/hr* releases a bomb. What is the horizontal distance the bomb has travelled when it hits the ground. (Take g=10 m/s^2). **3**

(d) (i) How many different ways could four cards be selected from a regular pack of 52 playing cards. **1**

(ii) How many of these selections will contain exactly two Aces. **1**

QUESTION 2 (9 Marks)

(a) A sky-diver opens his parachute when falling at 30 *m/s*. Thereafter, his acceleration is given by $\dfrac{dv}{dt} = k(6 - v)$, where k is a constant.

(i) Show that this condition is satisfied when $v = 6 + Ae^{-kt}$, and find the value of A. **2**

(ii) One second after opening his parachute, his velocity has fallen to 10.7 *m/s*. Find k to two decimal places. **2**

(iii) Find, to one decimal place, his velocity two seconds after his parachute has opened. **2**

(iv) If, with the same acceleration, the sky-diver opens his parachute when falling at 6 *m/s*, briefly describe his subsequent motion. **1**

(b) Persons A,B,C,D,E,F and G are to be seated at a round table. How many arrangements are possible if A refuses to sit next to B or C. **2**

QUESTION 3 (9 Marks)

(a) A particle moves in Simple Harmonic Motion. When it is 2 metres and 3 metres respectively from its centre of motion, its velocity is respectively 6 *m/s* and 4 *m/s*. Find the period of its motion and its amplitude. **3**

(b) A function *N(t)* is given by $N(t) = Ae^{\frac{t}{3}} + Be^{\frac{-2t}{3}}$, where A and B are constants.

(i) If *N(0)* = 30 and $N'(0) = -14$, find A and B. **2**

(ii) Find, to 2 decimal places, the value of t for which *N(t)* is a minimum, and find this minimum value. **3**

(iii) Briefly describe the behaviour of *N(t)* as t increases. **1**

QUESTION 4 (9 Marks)

(a) How many arrangements of the letters of the word CONTAINER are possible if:

(i) there are no restrictions. **1**

(ii) the vowels are together. **1**

(b) In a herd of 500 cows, the number N infected with a disease at time t years is given by $N = \dfrac{500}{1 + Ae^{-500t}}$.

(i) Briefly explain why all the cows will eventually be infected. **1**

(ii) Initially, only one cow was infected. After how many days will 200 cows be infected. **3**

(iii) Show that $\dfrac{dN}{dt} = N(500 - N)$. **3**

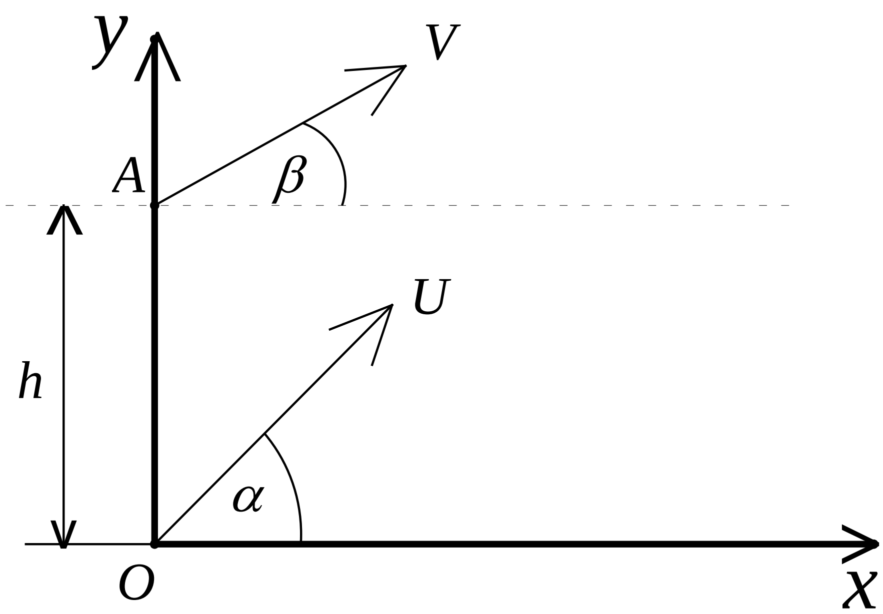
QUESTION 5 (9 Marks)

(a) The equation of motion of a particle moving in Simple Harmonic Motion is given by $x = a\cos(nt + \alpha)$, where x metres is its displacement from origin 0 after t minutes. It is initially 6 metres right of 0 and moving towards it. The period of its motion is 8 minutes and its maximum speed is 3π m/min. Find:

(i) the values of n, a and α. **3**

(ii) the first time when it passes through the origin. **1**

(b) In the diagram below, a particle is projected from the origin 0 with a speed of U m/s at an angle of elevation α .At the same instant, another particle is projected from A, h metres above 0 with a speed of V m/s at an angle of elevation β ($\beta < \alpha$) . The particles move in the same plane of motion and collide T seconds after projection. **5**



The horizontal and vertical components of displacement t seconds after the particle is projected from 0 are given by $x_o = Ut\cos\alpha$ and $y_o = Ut\sin\alpha - \frac{1}{2}gt^2$ respectively, and the horizontal and vertical components of displacement t seconds after the particle is projected from A are given by $x_A = Vt\cos\beta$ and $y_A = h + Vt\sin\beta - \frac{1}{2}gt^2$ respectively.

Show that $T = \dfrac{h\cos\beta}{U\sin(\alpha-\beta)}$.

QUESTION 6 (9 Marks)

(a) The displacement function of a particle moving x metres along a straight line after t seconds is given by $x = 3\cos^2 4t$. Show that its motion is Simple Harmonic and find its centre of motion. **3**

(b) The acceleration of a particle moving along a straight line is given by $\ddot{x} = 3x(x-2)$, where x metres is its displacement from the origin *0* after t seconds. Initially it is at *0* and its velocity is 2 *m/s* .

(i) Show that $v = 2(x^3 - 3x^2 + 2)$, where v is its velocity. **2**

(ii) Find its velocity and acceleration at $x = 1$. **2**

(iii) Briefly describe its motion after it moves from $x = 1$. **2**

QUESTION 7 (9 Marks)

(a) A velocity function is given by $\dfrac{dx}{dt} = (4-3x)^2$. Find $\dfrac{d^2x}{dt^2}$. **2**

(b) A team of FIVE is to be selected from a group of FOUR boys and FOUR girls.

(i) How many teams are possible if there is to be a majority of girls. **1**

(ii) What is the probability of a particular girl being included in the team and a particular boy not included, still assuming a majority of girls in the team. **2**

(c) On a certain day, the depth of water in a harbour at high tide is 11metres. At low tide $6\frac{1}{4}$ hours later, the depth of water is 7 metres. If high tide is due at 2.50 AM, what is the earliest time after midday that a ship requiring a depth of at least 10 metres of water can enter the harbour. **4**

Solutions to Year 12 Term 2 Assessment 2007

Question 1

(a)(i) SOCIETY

C						E

$5! = 120$

(ii) $p = \frac{5!}{7!} = \frac{1}{42}$

(b) $x = \sqrt{2}\cos 5t - \sin 5t$ —— (1)

$\dot{x} = -5\sqrt{2}\sin 5t - 5\cos 5t$

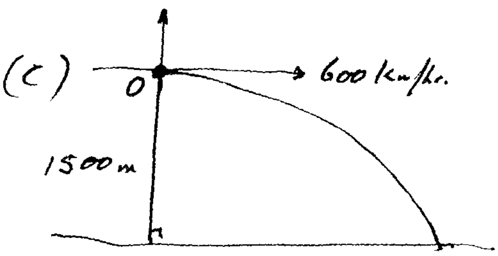
$\ddot{x} = -5^2\sqrt{2}\cos 5t + 5^2 \sin 5t$

$= -5^2(\sqrt{2}\cos 5t - \sin 5t)$

$\therefore \ddot{x} = -5^2 x$ from (1)

This is of the form $\ddot{x} = -n^2 x$ where $n = 5$.

(c)



$V = 600$ km/hr

$= 600 \times \frac{1000}{3600}$ m/s

$= \frac{500}{3}$ m/s

$\ddot{x} = 0$

$\dot{x} = V\cos\alpha$

$\therefore \dot{x} = V$

$\therefore x = Vt$

$\ddot{y} = -g$

$\dot{y} = -gt + V\sin\alpha$ ($\alpha =$

$\dot{y} = -gt$

$y = -\frac{gt^2}{2}$

Eliminating t:

$y = -\frac{gx^2}{2V^2}$

Taking O as origin, $g = 10$, $x = ?$, $y = -1500$ and $V = \frac{500}{3}$

$\therefore +\overset{300}{\cancel{1500}} = +\frac{10}{\cancel{2}}^{5} x^2 \div \left(\frac{500}{3}\right)^2$

$\therefore x^2 = \frac{500 \times 500 \times 100}{3}$

$= \frac{25 \times 10^6}{3}$

$\therefore x = \frac{5000\sqrt{3}}{3}$ metres or $\frac{5\sqrt{3}}{3}$ km.

Question 2

(a)(i) $v = 6 + Ae^{-kt}$

$\frac{dv}{dt} = -k \times Ae^{-kt}$

$= -k(v - 6)$

$\therefore \frac{dv}{dt} = k(6 - v)$

When $t = 0$, $v = 30$

$\therefore 30 = 6 + A$

$\therefore A = 24$

(ii) When $t = 1$, $v = 10.7$

$\therefore 10.7 = 6 + 24e^{-k}$

$\therefore \frac{4.7}{24} = e^{-k}$

$\therefore k = \ln \frac{24}{4.7}$

$\therefore k = 1.63$ (2 dec. pl.)

(iii) When $t = 2$

$v = 6 + 24e^{-2 \times 1.63}$

$\therefore v = 6.9$ m/s (1 dec. p

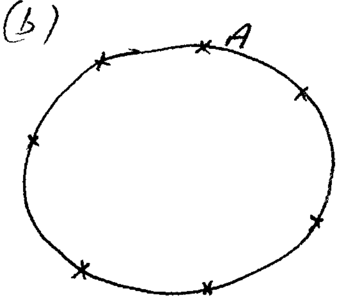
(iv) Acceleration is $\frac{dv}{dt} = k(6 - v)$. When $v =$

$\therefore \frac{dv}{dt} = 0$ $\therefore$ he fa

at a constant rat

of 6 m/s.

(b)



Seat A. Now B has 4 choices and C has 3. The remainder can be seated in 4! ways.

$\therefore 4! \times 4 \times 3$

$= 288$ arrangements.

Question 3

(a) Using $v^2 = n^2(a^2 - x^2)$

$\therefore 6^2 = n^2(a^2 - 2^2)$ and $4^2 = n^2(a^2 - 3^2)$

$\therefore 36 = n^2(a^2 - 4) \qquad 16 = n^2(a^2 - 9)$

$\therefore \dfrac{36}{a^2 - 4} = \dfrac{16}{a^2 - 9}$

$\therefore 36a^2 - 324 = 16a^2 - 64$

$\therefore 20a^2 = 260$

$\therefore a = \sqrt{13} \quad (a > 0)$

Let $4^2 = n^2(a^2 - 9)$

$\therefore 16 = n^2(13 - 9)$

$\therefore n = 2 \quad (n > 0)$

Now $T = \dfrac{2\pi}{n} = \dfrac{2\pi}{2}$

$\therefore T = \pi$ secs.

$\therefore$ the period of its motion is π secs. and its amplitude is $\sqrt{13}$ metres.

(b) $N(t) = Ae^{\frac{t}{3}} + Be^{-\frac{2t}{3}}$

(i)

$\therefore N'(t) = \dfrac{A}{3}e^{\frac{t}{3}} - \dfrac{2B}{3}e^{-\frac{2t}{3}}$

When $t = 0$

$A + B = 30$ —— (1)

$\dfrac{A}{3} - \dfrac{2B}{3} = -14$ —— (2)

(2): $A - 2B = -42$

$\therefore A - 2(30 - A) = -42$ from (1)

$\therefore A = 6$

$B = 24$

(ii) When $N'(t) = 0$, $\dfrac{A}{3}e^{\frac{t}{3}} = \dfrac{2B}{3}e^{-\frac{2t}{3}}$

$\therefore 2e^{\frac{t}{3}} = 16e^{-\frac{2t}{3}}$

$\therefore e^{t} = 8$

$\therefore t = 2.08$

When $t = 2.08$, $N = 6e^{\frac{2.08}{3}} + 24e^{-\frac{2}{3}\times 2.0}$

$\therefore N = 18$

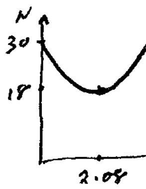
Test.

$N''(t) = \dfrac{A}{9}e^{\frac{t}{3}} + \dfrac{4B}{9}e^{-\frac{2t}{3}}$

at $t = 2.08$, $N''(t) = \dfrac{2}{3}e^{\frac{2.08}{3}} + \dfrac{32}{3}e^{-\frac{2}{3}\times}$

$= 3.99 > 0 \quad \therefore$ M

(iii) as $t \to \infty$, $e^{\frac{t}{3}} \to \infty$ and $e^{-\frac{2t}{3}} \to 0$. $\therefore N(t) \to \infty$.



Question 4

(a)(i) CONTAINER

$\dfrac{9!}{2!} = 181,440$

(ii) (AEIO) × × × × ×

$\dfrac{6!\,4!}{2!} = 8,640$

(b)(i) $N = \dfrac{500}{1 + Ae^{-500t}}$

As t increases, $e^{-500t} \to 0 \therefore N \to 500$

(ii) When $t = 0$, $N = 1 \therefore 1 = \dfrac{500}{1+A}$

$\therefore A = 499$

$\therefore N = \dfrac{500}{1 + 499e^{-500t}}$

When $N = 200$

$200 = \dfrac{500}{1 + 499e^{-500t}}$

$\therefore 1 + \dfrac{499}{e^{500t}} = \dfrac{5}{2}$

$\therefore e^{500t} = \dfrac{998}{3}$

$\therefore 500t = \ln \dfrac{998}{3}$

$\therefore t = 0.0116$ years

$\therefore t = 4.2$ days.

(iii) $N = 500(1 + Ae^{-500t})^{-1}$

$\therefore \dfrac{dN}{dt} = -500(1 + Ae^{-500t})^{-2} \times -500Ae^{-500t}$

$= \dfrac{500^2}{(1 + Ae^{-500t})^2} \times Ae^{-500t}$

$= N^2 \times Ae^{-500t}$

$= N^2\left(\dfrac{500}{N} - 1\right)$ since $1 + Ae^{-500t} = \dfrac{500}{N}$

$\therefore \dfrac{dN}{dt} = N(500 - N)$

Question 5

(a)(i) $x = a\cos(nt+\alpha)$

When $t=0$, $x=6$

$\therefore 6 = a\cos\alpha$

Since $T = \frac{2\pi}{n} = 8$

$\therefore n = \frac{\pi}{4}$ — (1)

Now $v^2 = n^2(a^2 - x^2)$

Since v is max. when $x=0$

$\therefore v^2 = n^2a^2 \quad \therefore v = \pm na$

Initially $x = +6 \quad \therefore v = +na$

$\therefore 3\pi = \frac{\pi}{4}a \quad \therefore a = 12$ — (2)

Now $6 = 12\cos\alpha \quad \therefore \cos\alpha = \frac{1}{2}$

$\therefore \alpha = \frac{\pi}{3}$ — (3)

Since $x = a\cos(nt+\alpha)$

$\therefore x = 12\cos\left(\frac{\pi t}{4} + \frac{\pi}{3}\right)$

When $x = 0$

$0 = 12\cos\left(\frac{\pi t}{4} + \frac{\pi}{3}\right)$

$\therefore \cos\frac{\pi}{2} = 12\cos\left(\frac{\pi t}{4} + \frac{\pi}{3}\right)$ by initial conditions

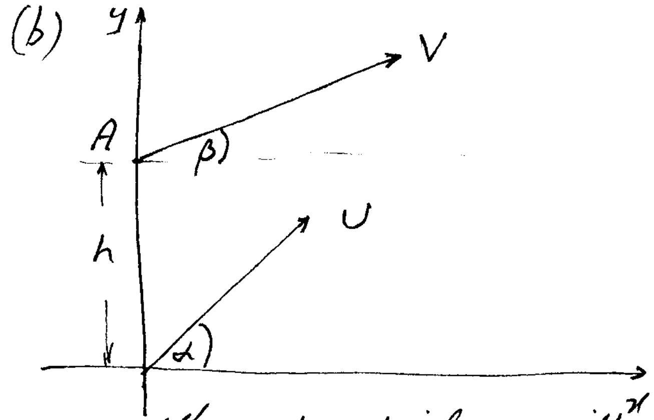
$\therefore \frac{\pi t}{4} + \frac{\pi}{3} = \frac{\pi}{2}$

$\therefore \frac{\pi t}{4} = \frac{\pi}{6}$

$\therefore t = \frac{2}{3}$

$\therefore$ it passes through the origin first time when $t = \frac{2}{3}$ minutes.

(b)



$x_o = Ut\cos\alpha$

$y_o = Ut\sin\alpha - \frac{1}{2}gt^2$

$x_A = Vt\cos\beta$

$y_A = h + Vt\sin\beta - \frac{1}{2}gt^2$

The particles will collide when $x_o = x_A$

$\therefore Ut\cos\alpha = Vt\cos\beta$

$\therefore U\cos\alpha = V\cos\beta$ —— (1)

When $t = T$, $y_o = y_A$

$\therefore UT\sin\alpha - \cancel{\frac{1}{2}gt^2} = h + Vt\sin\beta - \cancel{\frac{1}{2}gt^2}$

$\therefore T(U\sin\alpha - V\sin\beta) = h$

$$\therefore T = \frac{h}{U\sin\alpha - V\sin\beta}$$

$$= \frac{h}{U\sin\alpha - \frac{U\cos\alpha}{\cos\beta}.\sin\beta} \text{ from (1)}$$

$$= \frac{h}{\left(\frac{U\sin\alpha\cos\beta - U\cos\alpha\sin\beta}{\cos\beta}\right)}$$

$$\therefore T = \frac{h\cos\beta}{U\sin(\alpha-\beta)}$$

Question 6

(a) $x = 3\cos^2 4t$ — (1)

Now $\cos 8t = 2\cos^2 4t - 1$ — (2)

$\therefore \cos^2 4t = \frac{1}{2}(1+\cos 8t)$

$\therefore x = \frac{3}{2}(1+\cos 8t)$

$\therefore \dot{x} = -12 \sin 8t$

$\therefore \ddot{x} = -96 \cos 8t$

$= -96\left(\frac{2x}{3} - 1\right)$ from (1) and (2)

$\therefore \ddot{x} = -8^2\left(x - \frac{3}{2}\right)$

Since, of the form $\ddot{x} = -n^2(x - x_0)$ $\therefore$ SHM and centre of motion is at $x = \frac{3}{2}$, ie; $\frac{3}{2}$ metres right of origin.

Question 7

(a) $v = \frac{dx}{dt} = (4-3x)^2$

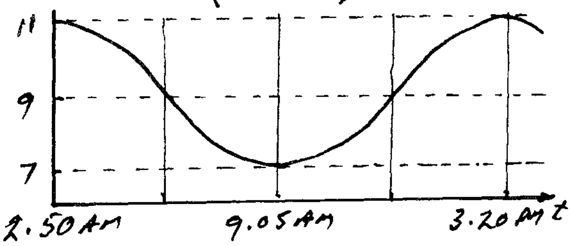
Now $\ddot{x} = \frac{d^2x}{dt^2} = \frac{d(\frac{1}{2}v^2)}{dx}$

$\therefore \ddot{x} = \frac{d\,\frac{1}{2}(4-3x)^4}{dx}$

$= \frac{4}{2}(4-3x)^3 \times -3$

$\therefore \ddot{x} = -6(4-3x)^3$

(c)



Let $x = a + b\cos(nt+\alpha)$

Since $a = 9$ and $b = 2$

$\therefore x = 9 + 2\cos(nt+\alpha)$

when $t = 0$, $x = 11$

$\therefore 11 = 9 + 2\cos\alpha$

$\cos\alpha = 1$ $\therefore \alpha = 0$

also $T = \frac{2\pi}{n} = 12\frac{1}{2} = \frac{25}{2}$

$\therefore n = \frac{4\pi}{25}$

(b) $\ddot{x} = v\frac{dv}{dx} = 3x(x-2)$

(i) $\therefore v\,dv = (3x^2 - 6x)\,dx$

$\therefore \int v\,dv = \int(3x^2 - 6x)\,dx$

$\therefore \frac{1}{2}v^2 = x^3 - 3x^2 + c$

when $x = 0$, $v = 2$ $\therefore c = 2$

$\therefore v^2 = 2(x^3 - 3x^2 + 2)$

(ii) At $x = 1$, $v^2 = 2(1-3+2)$

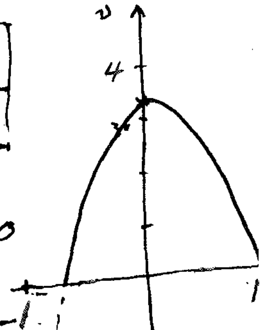
$\therefore v = 0$

and $\ddot{x} = 3(1-2)$

$= -3$ m/s^2

(iii)

x	-2	-1	0	1	2	3
v^2	-36	-4	4	0	-2	



at $x=1$, force is $-$ve,
$\therefore$ particle moves towards 0 with increasing speed.
at $x = 0$, particle slows down and stops between $x = 0$ and $x = -1$.
It oscillates between this point and $x = 1$, but not in SHM.

(b)(i) We require $2B + 3G$, $1B + 4G$

$\therefore {}^4C_2 \times {}^4C_3 + {}^4C_1 \times {}^4C_4$

$= 28$

(ii) $P = \frac{{}^3C_1 \times {}^3C_3 + {}^3C_2 \times {}^3C_2}{28}$

$= \frac{12}{18}$ or $\frac{3}{7}$

$\therefore x = 9 + 2\cos\frac{4\pi t}{25}$

When $x = 10$

$\therefore 10 = 9 + 2\cos\frac{4\pi t}{25}$

$\therefore \cos\frac{4\pi t}{25} = \frac{1}{2} = \cos\frac{\pi}{3}$

$\therefore \frac{4\pi t}{25} = 2n\pi \pm \frac{\pi}{3}$

Let $n = 1$ and consider $-\frac{\pi}{3}$

$\frac{4\pi t}{25} = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\therefore t = 10.42$ hrs or 10 hrs 25 minutes.

Now 2.50 AM + 10 hrs 25 min is 1.15 PM

$\therefore$ earliest time is 1.15 PM